Comment on: Thermal model for Adaptive Competition in a Market: Cavagna *et al.* [1] introduced an interesting model – called TMG as in [1] – similar to the minority game [2,3] (MG), of N agents interacting in a market. Strategies of agents are represented by D dimensional vectors $\vec{R}_i^a$ with $i = 1, \ldots, N$ running through agents and $a = 1, \ldots, s$ through i^{th} agent's available choices. The strategy $\vec{R}_i^\star$ used by i is selected drawing $a = \star$ from a Boltzmann distribution given by Eq. (4) of ref. [1] – or (4)-[1] for short – with "temperature" T and energies $-P(\vec{R}_i^a)$ (see Eq. (3)-[1]). Cavagna *et al.* [1] report numerical data showing an interesting collective behavior as a function of T (figs. 2-[1] and 3-[1]) and arrive at Eqs. (5,6)-[1] which are claimed to be the "exact dynamical equations for" the TMG. We show here that *i)* Eqs. (5,6)-[1] are incorrect *ii)* the correct continuum time dynamics is the same as that of the MG [4]. As a consequence the analytic solution of the MG of ref. [4] holds also for the TMG. Finally *iii)* the features found in [1] for $T \gg 1$ (figs. 2,3-[1]) are due to small simulation times and disappears if the system is in a steady state.

Cavagna *et al.* fail to define properly the continuum time limit (CTL) prescription, which is essential for stochastic differential equations such as Eq. (5)-[1]. It is crucial, in a proper derivation of the CTL, to observe that characteristic times in the TMG are proportional to D, as shown numerically in Fig. 1. This is natural because the adaptation of each agent's strategy requires an optimization of all its D components. This need sampling $\sim D$ values of $\vec{\eta}$, i.e. a time of order D. In order to eliminate the dependence of times on system size $N = D/d$, one has to rescale time by a factor D. The dynamics in the rescaled time $\tau = t/D$ is obtained iterating Eq. (3)-[1] from $t = D\tau$ to $D\tau'$

$$\frac{P(\vec{R},\tau') - P(\vec{R},\tau)}{\tau' - \tau} = \frac{-d}{D(\tau'-\tau)} \sum_{t=D\tau}^{D\tau'-1} A(t)\,\vec{R}\cdot\vec{\eta}(t). \quad (1)$$

The law of large numbers implies that, when $D = dN \to \infty$, the r.h.s. converges to $d\langle A\,\vec{R}\cdot\vec{\eta}\rangle$ where the average $\langle \ldots \rangle$ is both on the distribution π_i^a of $\vec{R}_i^\star$ *and* on that of $\vec{\eta}$. If we then let $\tau' \to \tau$ the l.h.s. converges to the derivative $\dot{P}$ of P w.r.t. τ. Hence, using Eq. (2)-[1] for $A(t)$ and $\langle \eta_\alpha \eta_\beta \rangle = \delta_{\alpha,\beta}/D$, Eq. (1) becomes $\dot{P} = -\frac{1}{N}\sum_i \langle \vec{R}_i^\star \rangle \cdot \vec{R}$ with $\langle \vec{R}_i^\star \rangle = \sum_a \pi_i^a \vec{R}_i^a$. The combination of Eq. (1) and Eq. (4)-[1] yields a dynamic equation for π_i^a, which reads

$$\dot{\pi}_i^a = -\frac{1}{NT}\pi_i^a \sum_{j=1}^{N} \langle \vec{R}_j^\star \rangle \cdot \left(\vec{R}_i^a - \langle \vec{R}_i^\star \rangle \right). \quad (2)$$

Eq. (2) coincides with the continuum time equation of ref. [4] which leads to the exact solution of the MG for $N \to \infty$. This depends only on the first two moments of the distribution of the components of $\vec{R}_i^a$, which plays the role of quenched disorder. Since, in the TMG, $\langle\langle \vec{R}_i^a \rangle\rangle = 0$ and $\langle\langle (\vec{R}_i^a)^2 \rangle\rangle = D$, these are the same as in the MG. Hence the two models have exactly the same collective behavior, as confirmed by Fig. 1.

Eq. (2) suggests that the dependence on T disappears by time rescaling. This is true in the $d \geq d_c$ phase: The T dependence for $T \gg 1$ reported in Figs 2,3-[1] is an artifact due to short simulation times (see also ref. [6]). For $d < d_c$ the CTL only holds for T larger than a crossover $T_c(d)$, as discussed elsewhere [5]. Indeed for $d = 0.1 < d_c$ and T large enough, data nicely collapses onto a single curve (see inset) once plotted against τ/T. For $T < T_c(d)$ the solution of Eq. (2) becomes dynamically unstable and the system enters into a turbulent regime where the CTL breaks down [5].

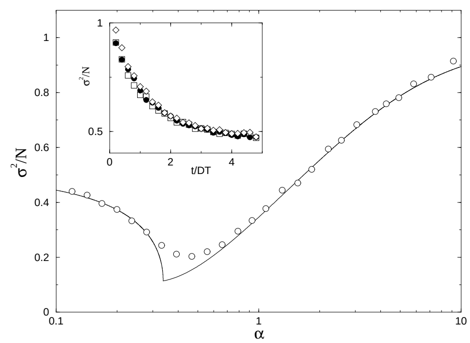

FIG. 1. σ^2/N as a function of d: numerical data with $S = 2$, $D = 64$ and $T = 10$ ($\circ$) and analytic solution [4] (full line). Finite size effects occur close to the phase transition $d \approx d_c$. Inset: relaxation of σ^2/N for $d = 0.1$, $DT = 10^5$ and $D = 25$ ($\bullet$) $D = 50$ ($\square$) and $D = 100$ ($\diamond$). Data collapse implies that characteristic times are proportional to D and T.

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